Due Sun

## 1.6 – More on Linear Systems and Invertible ~~Systems~~ Matrices

**Theorem 1.6.1** A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Pf: We know a system can have no solution or a unique solution.

If $A\vec{x}=\vec{b}$ has solutions $\vec{x}_1$ & $\vec{x}_2$, then

Let $\vec{x}_0 = \vec{x}_2 - \vec{x}_1$. Then $A(\vec{x}_2-\vec{x}_1) = A\vec{x}_2 - A\vec{x}_1$

$$= \vec{b} - \vec{b} = \vec{0}$$

If $k$ is any scalar, then

$$A(\vec{x}_1 + k\vec{x}_0) = A\vec{x}_1 + kA\vec{x}_0 = \vec{b} + \vec{0} = \vec{b}$$

Infinitely many choices for $k \Rightarrow$ infinitely many solutions exist.

**Theorem 1.6.2** If $A$ is an invertible $n \times n$ matrix, then for every $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

4. Solve the system by inverting the coefficient matrix and using Theorem 1.6.2.

$$5x_1 + 3x_2 + 2x_3 = 4$$
$$3x_1 + 3x_2 + 2x_3 = 2$$
$$x_2 + x_3 = 5$$

$$\left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 \to 3R_1 - 5R_2$$

we find that $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix}$ (verify on your own)

Then $\vec{x} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$

$$\boxed{\vec{x} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}}$$

(verify)

11. Solve the linear systems. Using the given values for the $b$'s solve the systems together by reducing an appropriate augmented matrix to reduced row echelon form.

$$4x_1 - 7x_2 = b_1$$
$$x_1 + 2x_2 = b_2$$

i. $b_1 = 0, b_2 = 1$

ii. $b_1 = -4, b_2 = 6$

iii. $b_1 = -1, b_2 = 3$

iv. $b_1 = -5, b_2 = 1$

$$\left[\begin{array}{cc|c} 4 & -7 & b_1 \\ 1 & 2 & b_2 \end{array}\right]$$

$R_2 \Rightarrow R_1 - 4R_2$

$$\begin{array}{ccc} 4 & -7 & b_1 \\ -4 & -8 & -4b_2 \\ \hline 0 & -15 & b_1 - 4b_2 \end{array}$$

$$\left[\begin{array}{cc|c} 4 & -7 & b_1 \\ 0 & -15 & b_1 - 4b_2 \end{array}\right]$$

$R_1 \Rightarrow 15R_1 - 7R_2$

$$\begin{array}{ccc} 60 & -105 & 15b_1 \\ 0 & 105 & -7b_1 + 28b_2 \\ \hline 60 & 0 & 8b_1 + 28b_2 \end{array}$$

$$\left[\begin{array}{cc|c} 60 & 0 & 8b_1 + 28b_2 \\ 0 & -15 & b_1 - 4b_2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2/15\, b_1 + 7/15\, b_2 \\ 0 & 1 & -1/15\, b_1 + 4/15\, b_2 \end{array}\right]$$

$$x_1 = \frac{2}{15} b_1 + \frac{7}{15} b_2$$
$$x_2 = -\frac{1}{15} b_1 + \frac{4}{15} b_2$$

i) $b_1 = 0,\ b_2 = 1$

$$\boxed{x_1 = \frac{7}{15},\ x_2 = \frac{4}{15}}$$

and so on.

17. Determine conditions on the $b_i$'s, if any, in order to guarantee that the linear system is consistent.

$$\begin{aligned} x_1 - x_2 + 3x_3 + 2x_4 &= b_1 \\ -2x_1 + x_2 + 5x_3 + x_4 &= b_2 \\ -3x_1 + 2x_2 + 2x_3 - x_4 &= b_3 \\ 4x_1 - 3x_2 + x_3 + 3x_4 &= b_4 \end{aligned}$$

rref for the augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & -8 & -3 & -b_1-b_2 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \end{array}\right]$$

is consistent if $b_1-b_2+b_3=0$ & $-2b_1+b_2+b_4=0$

$$\begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \\ \end{array}$$
$$\left[\begin{array}{cccc} 1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & -1 \end{array}\right] \begin{array}{l} \rightarrow b_1 = b_3+b_4 \\ \rightarrow b_2 = 2b_3+b_4 \end{array}$$

$$\boxed{\text{the system is consistent if } b_1=b_3+b_4 \ \& \ b_2=2b_3+b_4}$$

**Theorem 1.6.4** Equivalent Statements (extends Theorem 1.5.3)

If $A$ is an $n \times n$ matrix, then the following are equivalent.

a) $A$ is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of $A$ is $I_n$.

d) $A$ is expressible as a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

**Theorem 1.6.5** Let $A$ and $B$ be square matrices of the same size. If $AB$ is invertible, then $A$ and $B$ must also be invertible.

Pf: Let $\underline{\vec{x}_0}$ be a solution of $\underline{B\vec{x} = \vec{0}}$.

Then $(AB)\vec{x}_0 = A(\underline{B\vec{x}_0}) = A\vec{0} = \vec{0}$

So $\vec{x}_0 = \vec{0}$ (Thm 1.6.4 (a) & (b) applied to $AB$). So $B\vec{x} = \vec{0}$ has only the trivial solution $\Rightarrow$ $B$ is invertible

$$A = A(BB^{-1}) = \underset{\substack{\uparrow \\ \text{invertible}}}{\underline{(AB)}}\;\underset{\substack{\uparrow \\ \text{invertible}}}{B^{-1}}$$

Since $A$ is a product of 2 invertible matrices, $A$ is invertible.